

Principal Values and Weak Expectations

KENNY EASWARAN

University of Southern California

easwaran@usc.edu

This paper evaluates a recent method proposed by Jeremy Gwiazda for calculating the value of gambles that fail to have expected values in the standard sense. I show that Gwiazda's method fails to give answers for many gambles that do have standardly defined expected values. However, a slight modification of his method (based on the mathematical notion of the 'Cauchy principal value' of an integral), is in fact a proper extension of both his method and the method of 'weak expectations'. I show that this method gives an appropriate value when the 'tails' of the gambles that are eliminated in the truncation are 'stable', but that the value is not appropriate when the tails are not stable. I do not attempt to give an argument for the use of this method, but just note that it is more general than Gwiazda's method, and is mathematically quite natural.

1. Introduction

In his 2014, Jeremy Gwiazda proposes a method to calculate the value of a gamble that has infinitely many possible outcomes. (I will define this method and various others for calculating values later on in the paper. These definitions are summarized at the end in Sect. 10.) His method is proposed as a supplement to the notion of a 'weak expectation' developed by Easwaran (2008). Along the way, he rejects various other equally natural-seeming methods, including one that he terms the 'mechanistic fallacy'. I will not focus on the arguments against these alternatives (some of which were already given by Nover and Hájek (2004), and some of which are original to Gwiazda, but all of which seem decisive). In this paper, I will instead investigate Gwiazda's positive proposal. I will show that, as stated, it is substantially weaker than one might think, but that it can be developed by taking the 'Cauchy principal value'. I will consider the conditions under which this method of evaluating gambles is well defined and show that it extends both Gwiazda's method, and the method of weak expectations (which is itself an extension of other standard methods for valuing a gamble), but is itself undefined for additional gambles. I will not consider arguments

for or against the use of this method—evaluation of the normative force of this proposal is a topic for future investigation.

Throughout I will primarily consider ‘discrete’ gambles, meaning that there are only countably many possible outcomes of the gamble, each of which has a real-valued utility and a positive real-valued probability, and these probabilities sum to 1. (I will briefly mention some ‘continuous’ gambles, with uncountably many possible outcomes, towards the end—the points I make are far more pressing in that context.)

Gwiazda’s investigation presupposes that the value of a discrete gamble is some sort of sum of the probability-weighted utilities of the outcomes (that is, the utility of each outcome times its probability). If there are only finitely many outcomes, then the sum of the probabilities (required for the definition of a discrete gamble) and the sum of probability-weighted utilities are both unproblematic. However, when there are infinitely many outcomes, Gwiazda notes that there are problems. There is a standard definition of a sum of infinitely many terms when these terms are indexed by the natural numbers. In that case, we say that the sum of the infinitely many terms is the limit as n goes to infinity of the finite sum of the first n terms, if this limit exists, and the sum is undefined otherwise. Gwiazda therefore concentrates on finding a way to assign a unique order to the outcomes of a gamble, so that we can sum their probability-weighted utilities.

I will show that coming up with an order to compute this sum is not generally an appropriate way to figure out the value of discrete gambles with infinitely many possible outcomes, and will demonstrate a different process to take the limit of. This other process is both far more general than Gwiazda’s, and is a standard technique in certain mathematical applications, and thus seems like a more natural method to focus on.

2. Summing infinitely many terms

When a set of terms are all positive (like the probabilities of the outcomes) or all negative, it turns out that the order does not matter—for *any* indexing of the terms by natural numbers, the infinite sum of the terms is always the same. Thus, there is no problem with the condition in the definition of a discrete gamble that the probabilities sum to 1. If all outcomes have positive utility, or all outcomes have

negative utility, then there is also no problem in computing the sum of the probability-weighted utilities, and this is often taken to be the value of the gamble.

When a set of terms includes both positive and negative terms, there are still conditions under which we do not have to worry about the order. The set of positive terms has a well-defined sum, and so does the set of negative terms. If one or both of these sums is finite, then again, any indexing of the terms by natural numbers gives the same infinite sum, which is just the result of adding together the sum of the positive terms and the sum of the negative terms.

Conveniently, if there is some number U such that every outcome of the gamble has utility less than U , then the set of positive probability-weighted utilities has a finite sum.¹ Similarly, if there is a lower bound for the negative utilities of the outcomes, then there is also a well-defined sum. Of course, it is not necessary that the outcomes have bounded utility in order for the positive or the negative sum to be well-defined and finite, but it is sufficient.

If the sum of the positive terms and the sum of the negative terms are both infinite, then the sum of all the terms is not uniquely determined by the set of terms—it is this case that drives Gwiazda to look for an order for the terms. If there is some ϵ such that infinitely many of the terms in the sum have absolute value greater than ϵ , then it is clear that no indexing of the terms gives rise to a sequence of partial sums with a finite limit (because convergence to a finite limit requires the sequence of partial sums to eventually stabilize within an interval of size less than ϵ). But if there is no such ϵ , then the Riemann Rearrangement Theorem guarantees that for every real number x (and also for $+\infty$ and $-\infty$), there is an indexing of the terms such that the sequence of partial sums converges to x .

Thus, in this sort of case, where positive utilities and negative utilities are both unbounded, and the sum of the probability-weighted utilities is infinite on both sides, if we want to say that the value of a gamble is the sum of the probability-weighted utilities, then we need to find a way to index the outcomes by natural numbers, in order to compute the sum.

¹ Proof: consider any indexing of the outcomes as o_n , each with utility u_n and probability p_n . The partial sums of the positive terms $u_n \cdot p_n$ form an increasing sequence, and are always less than the infinite sum of $U \cdot p_n$, which is U . An increasing sequence of real numbers that is bounded from above must converge to a number no greater than the bound, which is thus finite.

3. Gwiazda's method

Gwiazda proposes indexing the outcomes in increasing order of the absolute value of the utility associated with the outcome, after properly rejecting indexes based on the mechanism that generates the gamble, or the probabilities of the outcomes. While this method works for the gambles he considers, for many other gambles it does not give an indexing by natural numbers. There may be some outcomes such that infinitely many others come before it on this ordering.

As an example, consider gamble *A*, which has the following outcomes. For each positive integer n , there is an outcome whose probability is $1/2^{n+1}$ and whose utility is $1 - 1/n$. Additionally, for each positive integer n , there is an outcome whose probability is $1/2^{n+1}$ and whose utility is $2 - 1/n$. The outcomes of the first type have probabilities that sum to $1/2$, and the outcomes of the second type also have probabilities that sum to $1/2$, so this is in fact a discrete gamble.

However, if we order the outcomes by increasing absolute value of utility, then the infinitely many outcomes of the first set (whose absolute values are all less than 1) will all come before any outcome of the second set (whose absolute values are all at least 1). To index them in this order, we would have to resort to something like Cantor's ordinals — the terms of the first set can be indexed by the standard natural numbers, and the terms of the second set can be indexed by the ordinals $\omega, \omega + 1, \omega + 2, \dots$

There might be a natural way to define a sum over this sort of non-standard order type, by summing the first set of terms, and then adding the sum of the second set, but for other gambles the problem can be much worse. Consider gamble *B*, which has, for each positive integer n , an outcome whose probability is $1/2^n$ and whose utility is $1/n$. In this gamble, *every* outcome has infinitely many other outcomes with smaller utility. Thus, there are *no* finite initial segments of the sequence of outcomes in this ordering, and Gwiazda's ordering does not give a way to compute the sum.

Even if one found a way to sum this particular set of terms with its unusual order, there are further order-types to consider. In fact, for *every* countable order type, there is a set of real numbers that has this order type (in fact, the rational numbers within any finite interval are sufficient to contain a set of any countable order type), which can be taken to be the utilities of the outcomes of some gamble. To calculate

the sum of the probability-weighted utilities in an ordering by absolute value of the utilities, one would need a general method of summing over arbitrary countable orderings, which is exactly the problem that Gwiazda tried to address with the utilities of the outcomes. Indexing based on absolute value of utility gives an indexing by natural numbers for some gambles, but does nothing at all to help for plenty of other gambles. Gwiazda's method gives a value for each of the gambles he considers, but very many discrete gambles cannot be evaluated at all by his method.

4. Expectation without summation

In fact, for all the gambles mentioned so far, there is a way to calculate their expectation without taking an infinite sum of the sort that Gwiazda considers. Since the utilities of the outcomes are bounded, any way of indexing the outcomes by natural numbers and summing the probability-weighted utilities in this order will give the same value. This value is said by measure theorists to be the 'expectation' of the gamble, and is called the 'strong expectation' by Easwaran (2008). But the general way to calculate this value is not as a single sum. I will first define a value that I will write as $E(X)$, and then show how to calculate the strong expectation of a gamble in terms of this.

When X is a 'simple' gamble (that is, a discrete gamble with only finitely many possible outcomes), $E(X)$ is defined (as Gwiazda suggests) to be the finite sum of probability-weighted utilities. But for gambles with infinitely many possible outcomes, its definition is slightly more complex. A gamble X is said to 'stochastically dominate' a gamble Y iff, for every x , the total probability that X gives the set of outcomes with utility at least x is at least as high as the total probability that Y gives the set of outcomes with utility at least x . Stochastic dominance is a way to compare gambles that have completely different sets of outcomes, but where one gives higher probability for good outcomes than the other. If X is a gamble whose outcomes have a finite lower bound, then $E(X)$ is defined to be the smallest real number x such that for any simple gamble X' that is stochastically dominated by X , $x \geq E(X')$. (This is written as 'sup $E(X')$ ', for the 'supremum' over these values.) It is not hard to see that for discrete gambles, this definition always gives the same value as the summation of the probability-weighted utilities in any indexing by the natural numbers, even though it is calculated as a limit of expectations of the X' , rather than as the sum over the outcomes of X in some order or other.

This definition is then extended to gambles with a finite upper bound on the utility of their outcomes by reversing all the comparisons— $E(X)$ is defined to be the greatest real number x such that for any simple gamble X' that stochastically dominates X , $x \leq E(X')$ (written as ‘ $\inf E(X')$ ’, for the ‘infimum’ over these values). For gambles with both finite upper and lower bounds, it is not hard to check that these two definitions coincide.

If X has outcomes that are unbounded in both the positive and the negative direction, then there is no simple gamble that either dominates or is dominated by X , so $E(X)$ is not defined. However, if we let X^+ be the gamble that is like X but replaces all outcomes whose utility is negative by outcomes whose utility is 0, and X^- be the gamble that replaces all outcomes whose utility is positive by outcomes whose utility is 0, then $E(X^+)$ and $E(X^-)$ are both defined. As long as at least one of these values is finite, we can then define the strong expectation of X to be $E(X^+) + E(X^-)$.

These definitions of $E(X)$ and of strong expectation in terms of a supremum and infimum, rather than a sum, apply perfectly well for gambles A and B , for which Gwiazda’s method does not provide an order for calculating the sum.² Thus, if we take Gwiazda’s suggestion literally, then there are many gambles that have strong expectations, but for which his suggestion is undefined.³ But I will show that we can preserve Gwiazda’s insight of starting with outcomes of relatively small absolute value while working outwards, and extend it by means of the more general definition of expectation given here. This will yield a method that is a strict generalization of Gwiazda’s method, as well as of the methods of strong expectation and of weak expectation.

² This is particularly clear in the continuous case, where these definitions correspond to a notion of an integral rather than a sum. But A and B show that even when limiting to discrete gambles, there are important restrictions to Gwiazda’s method.

³ Similarly problematic gambles can be arranged for the ordering Gwiazda considers by decreasing probability-weighted utility.

Interestingly, this problem does not arise for the index by decreasing probability. For any outcome o in any gamble X , there can be at most finitely many outcomes of X whose probability is higher than the probability of o , since o has positive probability, and the probabilities of the outcomes must sum to 1. Of course, the other problem for this ordering mentioned by both Gwiazda and Hájek (that two gambles that have the same probabilities of producing all the same utilities can end up being summed in arbitrarily different orders) seems to be a different decisive argument against using this ordering.

5. Principal values

If X is any gamble, and n is any positive real number, then let X_n be the gamble that is ‘truncated’ at n . That is, for every outcome of X whose utility has absolute value at most n , X_n has an outcome with the same probability and utility. However, X_n has no outcomes whose utility has absolute value greater than n . Let $p_n(X)$ be the total probability of all outcomes of X whose utility had absolute value greater than n . X_n replaces all these outcomes by a single outcome with utility 0 and probability $p_n(X)$.

Each X_n is thus bounded, and therefore has a well-defined expectation $E(X_n)$. We can then consider the limit of $E(X_n)$ as n goes to infinity, and use this as a value for X itself. When the outcomes in X can in fact be indexed in order of increasing utility, then Gwiazda’s sum will coincide with this limit (because each $E(X_n)$ will correspond to some partial sum in the resulting infinite series). But this process preserves the spirit of Gwiazda’s suggestion while extending it to many more gambles, including every gamble that has a well-defined strong expectation. I will call this the ‘principal value’ of the gamble, following the standard mathematical definition of the ‘Cauchy principal value’ of an improper integral.

In comparing how similar X and X_n are, it is natural to look at the ‘tails’, which consist of all outcomes of X whose utilities have absolute value greater than n . Then we can say the ‘size’ of the tail is $n \cdot p_n(X)$. It turns out that whether the limit of $E(X_n)$ makes sense as a value for X depends on the behaviour of these tails.

6. Stable values

Hájek (2013, pp. 9–10) notes that utility is only defined up to a shift and a stretch — there is no well-defined 0 and no well-defined unit in which utility is measured. Thus, it ought to be the case that adding a constant to the utility of every outcome of a gamble affects the overall value of the gamble by adding the same constant, and similarly for multiplying the utilities by a positive constant. I will show that the principal value always has the second property, and it has the first property iff the tails have a feature I will call ‘stability’. To illustrate this feature, I will demonstrate a gamble whose tails are not stable, and show that for this particular gamble, adding a constant to every utility actually does change the principal value in a more complicated way.

Thus, the method proposed here is to evaluate a gamble at its principal value, iff the gamble has stable tails.

The rest of this section consists of proofs of the relevant facts, and can be skipped by readers who are not interested in the mathematical details. A similar result, which characterizes the concept of stability in a different way, is proven by Elmer et al. (MS). In what follows, all limits will be as n goes to infinity, and I will, without loss of generality, let k be an arbitrary positive constant.

First, I will show that changing the unit of utility by a factor k while keeping o fixed has the expected effect on the principal value. Let X be any gamble, and let X' be a gamble whose outcomes correspond to those of X , with the same probability, but with utility that is k times the utility of the corresponding outcome. Let X'_n be the truncation of X' , just as X_n is the truncation of X . It is straightforward to see that $E(X'_{kn}) = kE(X_n)$, and thus $\lim E(X'_n) = k \lim E(X_n)$, if either limit exists.

Now I turn to the case of shifting the zero point for utility by k units. Let X be any gamble, and let X' be a gamble whose outcomes correspond exactly to the outcomes of X , with the same probability, but with utility that is k units higher than the utility of the corresponding outcome of X . Let X'_n be the truncation of X' , just as X_n is the truncation of X . $p_n(X')$ is the tail probability of X' , just as $p_n(X)$ is the tail probability of X .

Assume that $E(X_n)$ converges to a finite value. Then $\lim E(X'_n) = k + \lim E(X_n)$ iff $(E(X'_n) - k) - E(X_n)$ converges to 0, or equivalently $E(X_n) - (E(X'_n) - k)$ converges to 0. Since $E(X_n)$ converges to a finite value, $E(X_n) - E(X_{n+k})$ and $E(X_{n-k}) - E(X_n)$ both converge to 0. By additivity of limits, it thus suffices to check that either $(E(X'_n) - k) - E(X_{n-k})$ or $E(X_{n+k}) - (E(X'_n) - k)$ converges to 0, in which case both must.

First, consider the limit of $(E(X'_n) - k) - E(X_{n-k})$. $E(X'_n)$ is the sum of the probability-weighted utilities of all outcomes of X' whose utility is between $-n$ and n . Since each of these outcomes has utility k more than some corresponding outcome of X , if we subtract k from this, then we get the sum of the probability-weighted utilities for the corresponding outcomes of X (all the ones whose utility is between $-(n+k)$ and $(n-k)$) together with $-k \cdot p_n(X')$. Subtracting $E(X_{n-k})$ means that we cancel out the probability-weighted utilities for the outcomes of X whose utility is between $-(n-k)$ and $(n-k)$, leaving the sum of the probability-weighted utilities of outcomes of X whose utility is between $-(n+k)$ and $-(n-k)$, together with

$k \cdot p_n(X')$. Since we are taking limits, and $p_n(X')$ goes to 0, we can ignore this last term. Thus, in the limit, $(E(X'_n) - k) - E(X_{n-k})$ is the sum of the probability-weighted utilities of the outcomes of X whose utility is between $-(n+k)$ and $-(n-k)$.

It is not obvious when the limit of this sum is 0. But by the earlier considerations, we know that the limit of this sum is 0 iff the limit of $E(X_{n+k}) - (E(X'_n) - k)$ is 0. And a similar calculation shows that in the limit, this is the sum of the probability-weighted utilities of the outcomes of X whose utility is between $(n-k)$ and $(n+k)$.

Two sequences both converge to 0 iff the sum of their absolute values converges to 0. This is the sum of the probability-weighted absolute values of the utilities of all outcomes of X whose absolute value is between $(n-k)$ and $(n+k)$. Since the total probability of these outcomes is $p_{n-k}(X) - p_{n+k}(X)$, we can thus see that the sum of the absolute values is between $(n-k)(p_{n-k}(X) - p_{n+k}(X))$ and $(n+k)(p_{n-k}(X) - p_{n+k}(X))$. Since k is a constant, and $p_n(X)$ converges to 0, each of these values converges to 0 iff $(n-k)p_{n-k}(X) - (n+k)p_{n+k}(X)$ does. Thus, if the limit of $E(X_n)$ is defined and finite, then the limit of $E(X'_n)$ is defined and equal to it iff $\lim((n-k)p_{n-k}(X) - (n+k)p_{n+k}(X)) = 0$.

This is the condition I will call 'stability'. I will say that X has stable tails iff there is some positive k such that

$$\lim((n-k)p_{n-k}(X) - (n+k)p_{n+k}(X)) = 0$$

It is straightforward to see that this holds for some positive k iff it holds for all positive k , since the individual terms of the sequence for large k just correspond to the sum of several successive terms of the sequence for a smaller k . An easier condition to check that entails this condition is that $\lim n \cdot p_n(X)$ exists and is finite. (This condition is strictly weaker since it can fail while stability holds if the $n \cdot p_n(X)$ form some sequence that increases without bound while successive terms get closer and closer together.)

To illustrate the stability condition, consider the gamble C , defined as follows. For each positive integer m , there is an outcome of C whose probability is $1/2^{m+1}$ and whose utility is 2^m . Additionally, for every positive integer m , there is an outcome of C whose probability is $1/2^{m+1}$ and whose utility is $-(2^m)$. (This is a mixture of the St. Petersburg gamble with its negative.) The truncation C_n is always symmetric, and has expectation equal to 0. Thus, the principal value of this gamble is 0.

For this gamble, the tails have probability $p_n(C)$ where $p_n(C) \approx 2/n$ if n is just short of a power of 2, and $p_n(C) \approx 1/n$ if n is just above a power of 2. Thus, if n is a power of 2, then no matter how small k is, we have $(n - k) \cdot p_{n-k}(C) - (n + k) \cdot p_{n+k}(C) \approx 1$, so these terms do not converge to 0. And indeed, if C' is the same gamble as C , but with the utilities of every outcome increased by 1, then the limit of $E(C'_n)$ is undefined. If n is just short of a power of 2, then C'_n will include a negative outcome that is not matched by any corresponding positive outcome, and will have expected value $1/2$. If n is slightly more than $2^m + 1$, then the negative and positive outcomes will balance, and it will have expected value 1. Thus, there is no limit of $E(C'_n)$, demonstrating the sensitivity to shifts of scale. (In this case, Gwiazda's proposal of summing in order by utility gives a non-convergent series $1/2 - 1/2 + 1/2 - 1/2 + \dots$ — I believe this will happen in general, so that Gwiazda's proposal will give a value only if the tails are stable.)

7. Principal values vs. weak expectations

Gwiazda's proposal fails to assign a value to many gambles that have a well-defined strong expectation. The principal value is a natural extension of Gwiazda's proposal that does assign a value to every gamble that has a strong expectation, and in fact agrees with the strong expectation in these cases. I will now compare this principal value to the value assigned by weak expectations.

The proof given by Easwaran (2008) shows that for any gamble X , if $\lim n \cdot p_n(X) = 0$, and the principal value is well-defined and finite, then it is in fact equal to the weak expectation of X . Furthermore, as noted by Durrett (2005, p. 41) (and proved in Feller 1971, pp. 235–6), if there is *any* sequence of values a_n such that the average of n plays of X converges to a_n (the case of X having a weak expectation is the case where the a_n are constant), then $\lim n \cdot p_n(X) = 0$, and the a_n and the $E(X_n)$ must converge to each other. Thus, if a weak expectation exists, then it is equal to the principal value, and no weak expectation exists unless $\lim n \cdot p_n(X) = 0$. If the limit is 0, I will say that the tails are 'thin'.

Thus, as long as there is a gamble such that the principal value is defined, and whose tails are stable but not thin, then this new method does indeed extend the method of weak expectations.

And in fact, such a case exists. The gamble can be built with the following tools. Consider the terms $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. We will use these

terms to build a gamble that has a limit of truncated expectations, but which has stable tails that are not thin. Consider the sum of all terms of this form as n goes from N to infinity. $\sum_{n=N}^{\infty} \frac{1}{n(n+1)} = (\frac{1}{N} - \frac{1}{N+1}) + (\frac{1}{N+1} - \frac{1}{N+2}) + \dots$. This is a ‘telescoping’ series, where parts of adjacent terms cancel. Thus, it is not hard to see that the sum of the first k terms in this series is $\frac{1}{N} - \frac{1}{N+k}$, and thus the sum of the series is the limit of these partial sums, which is $1/N$. In particular, the sum from 1 to infinity is 1.

Define a gamble D such that for each positive integer n , there is an outcome whose probability is $\frac{1}{n(n+1)}$, and whose utility is $(-1)^{n+1} \cdot (n+1)$. These probabilities sum to 1, so this is a well-defined gamble.

Then for any positive integer N , $p_N(D)$ is $\sum_{n=N+1}^{\infty} \frac{1}{n(n+1)} = 1/(N+1)$. Therefore, the limit of $N \cdot p_N(X)$ is the limit of $N/(N+1)$, which is 1. Thus, the tails are not thin, so there is no weak expectation, but the tails are stable, so the principal value interacts appropriately with changes of scale for utility.

$E(D_n) = (2 \cdot 1/2) + (-3 \cdot 1/6) + \dots + ((n+1) \cdot (-1)^{n+1} \cdot 1/n(n+1))$, which is the sum

$$1 - 1/2 + 1/3 - \dots + (-1)^n/(n+1)$$

This is an alternating harmonic series, and its limit is $\log 2$, which is thus the value of the game according to the principal value method. (This gamble is exactly the ‘Arroyo game’ introduced by Bartha (MS) to demonstrate that there are gambles whose sum of probability-weighted utilities converge under certain arrangements, but for which the weak expectation is undefined.)

8. Alexander on the Cauchy distribution

An example of a continuous gamble (as opposed to the discrete ones considered earlier) with the relevant features is the Cauchy distribution, discussed at length in Alexander 2012. Its probability density for each utility x is given by $f(x) = \frac{1}{\pi(1+x^2)}$. This gamble is symmetric, and thus every truncation is symmetric as well, so for any n , $E(X_n)$ is 0. It turns out that as n goes to infinity, the limit of $n \cdot p_n(X)$ is $\frac{2}{\pi}$.⁴

⁴ One can see this by noting that for large x , $\frac{1}{\pi(1+x^2)} \approx \frac{1}{\pi x^2}$, and thus for large n the tail probability p_n is approximately $\int_n^{\infty} \frac{1}{\pi x^2} dx = \frac{2}{\pi n}$.

Thus, the tails are stable, so the principal value behaves appropriately with changes in the scale of utility.

However, as Alexander points out, this distribution has no weak expectation. The weak expectation is defined as that value, if any, to which the average of n independent copies of a gamble converges in probability, as n goes to infinity. However, the average of n independent copies of the Cauchy distribution is in fact the Cauchy distribution itself. Averaging large numbers of plays does not make the distribution converge to a single value, and thus there is no weak expectation. (One could also just observe that the tails are not thin, but Alexander's argument is more direct for this specific case.)

In section two of his paper, Alexander explicitly considers the principal value as a way to evaluate the Cauchy distribution, but rejects it. He suggests that once we consider the truncation X_m , we should also consider asymmetric truncations X_{-m}^n , which include all outcomes whose utility is between $-m$ and n , where m and n might not be equal. He shows that, for the Cauchy distribution, $E(X_{-n}^n)$ always equals 0, but that by choosing m and n properly we can arrange for $E(X_{-m}^n)$ to achieve any value, positive or negative. If the truncations do not grow outwards at the same rate in the positive and negative direction, then the limit can be any value. He thinks that limiting consideration to symmetric truncations is an arbitrary restriction, and thus rejects the principal value as a way to extend weak expectation to further gambles.

In fact, this problem of asymmetric limits applies to any gamble that fails to have a strong expectation. A gamble has a strong expectation iff either the positive part or the negative part (or both) has a finite expectation. But if both of these conditions fail, then by choosing appropriate asymmetric truncations, one can do the trick that leads to the Riemann Rearrangement Theorem and get the expectations of the truncations to converge to any value one wants. Thus, this challenge would apply even in cases where there is a weak expectation, and not just to extensions of the method.

I do not claim to have a response to this charge of arbitrariness, but I note it here. The same worry applies to the discrete gamble D , so it is not just a feature of continuous gambles. A proper justification of the method of principal values would have to answer Alexander's charge. But this charge applies just as well to Gwiazda's method of summing in order of absolute value of utility, since this method makes the same assumption of symmetry. Since weak expectations have a characterization in terms of behaviour under repeated play, perhaps they can

avoid this charge of arbitrariness. And if so, perhaps some similar alternate characterization can justify the use of the principal value even in cases where the tails are stable but not thin.

9. Conclusion

Gwiazda's recommendation fails to give a value to gambles whose outcomes are not well-ordered by increasing absolute value of utility, many of which have well-defined strong expectations. The principal value, defined as the limit of the expectations of the truncations $E(X_n)$, is always equal to the sum given by Gwiazda's method when it gives an answer, but it is also defined in many circumstances that Gwiazda's method is not. In particular, it always agrees with the weak expectation (and thus the strong expectation) whenever it exists, but it sometimes exists even when the weak expectation does not, so it is in fact a generalization. The principal value behaves appropriately under changes of the scale on which utility is measured iff the tails are stable, so this gives a reasonable restriction for the cases in which it is appropriate to use. But even with this restriction, this is still a proper generalization of the notion of weak expectation.

Alexander claims that there is an arbitrariness to truncating gambles at the same distance on the positive and negative side of the utility scale. If this is right, then perhaps we should retreat to the method of weak expectations, but if there is a good argument for this symmetry, then Gwiazda's initial suggestion has led to a useful extension of the notion of weak expectations.⁵

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⁵ A summary of definitions and results is given in an appendix below.

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Appendix: Summary of definitions and results

Gwiazda's expectation of X : Sum of probability-weighted utilities of all outcomes of X , calculated in order of increasing absolute value of utility of the outcome, if this is a well-ordering and the sum converges.

$E(X)$: Sum of probability-weighted utilities of all outcomes of X , if the utilities of the outcomes have a finite upper or lower bound (or both). Given as a supremum or infimum for gambles that are not discrete, as in section 4.

X^+ : Gamble that includes all outcomes of X whose utility is positive, and a single new outcome whose utility is 0 and whose probability is the sum of the probabilities of outcomes of X that are non-positive.

X^- : Gamble that includes all outcomes of X whose utility is negative, and a single new outcome whose utility is 0 and whose probability is the sum of the probabilities of outcomes of X that are non-negative.

Strong expectation of X : $E(X^+) + E(X^-)$, provided that at least one of these is finite.

$p_n(X)$: Probability that the outcome of X has absolute value greater than n .

X_n : Gamble that includes all outcomes of X whose utility is between $-n$ and n , and a single new outcome whose utility is 0 and whose probability is $p_n(X)$.

X has thin tails: $\lim_{n \rightarrow \infty} n \cdot p_n(X) = 0$.

Weak expectation of X : $\lim_{n \rightarrow \infty} E(X_n)$, if X has thin tails.

X has stable tails: $\lim_{n \rightarrow \infty} ((n - k) \cdot p_{n-k}(X) - (n + k) \cdot p_{n+k}(X)) = 0$ for some (equivalently, all) positive k .

Principal value of X : $\lim_{n \rightarrow \infty} E(X_n)$, if X has stable tails.

If Gwiazda's expectation of X exists then it is the principal value of X . If $E(X)$ exists then it is the strong expectation of X . If the strong expectation of X exists then it is the weak expectation of X . If the weak expectation of X exists then it is the principal value of X . None of the converses holds.

At each level of this hierarchy (principal value but no weak expectation, weak but no strong expectation, strong expectation but no $E(X)$), some discrete gambles have a Gwiazda's expectation and some do not. Only discrete gambles have a Gwiazda's expectation, while the other definitions all apply to some continuous gambles.